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DOUBLY PERIODIC MAXIMAL SURFACES WITH SINGULARITIES

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Abstract

We consider almost entire solutions to the maximal surface equation which satisfy certain structure conditions. This approach allows us to construct a large class of periodic solutions by using generating matrices. We describe the analytic behavior of such solutions with mixed-type singular points.

Key words and phrases: maximal surface, doubly periodic maximal surface, singularity.

1. Introduction

1.1. This article is devoted to studying almost entire doubly periodic solutions to the following nonlinear equation:

$$\left(1-f_y'^2\right)f''_{xx}+2f'_xf'_yf''_{xy}+\left(1-f_x'^2\right)f''_{yy}=0. \tag{1}$$

It is well known that, outside the set Σ_f of points where the norm of the gradient of the function $f(x,y)$ equals 1, the graph $z=f(x,y)$ of a solution to (1) represents a surface of mean curvature zero in the Minkowski space $\mathbb{R}^3_1(x,y,z)$ endowed with the indefinite metric

$$ds^2=dx^2+dy^2-dz^2.$$

As follows from the results by Calabi [1] and Cheng and Yau [2], every entire solution to (1), i.e. a solution defined everywhere in $\mathbb{R}^2$, is an affine

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function $f(x,y) = ax + by + c$, provided that the extra requirement

$$|\nabla f(x,y)| < 1 \tag{2}$$

is satisfied. Inequality (2) may be interpreted as the space-like condition for the graph $z = f(x,y)$ in $\mathbb{R}^3_1$. In this case, equation (1) can be written down in the divergence form

$$\operatorname{div} \frac{\nabla f}{\sqrt{1 - |\nabla f|^2}} = 0 \tag{3}$$

and is referred to as the maximal surface equation. Thus, every solution to the maximal surface equation (3) (or problem (1), (2)) which is defined over the whole plane $\mathbb{R}^2$ must have singularities whenever it is not an affine function.

We call a function $f(x,y)$ of the class C^2 an *almost entire solution* to (1) if it is defined everywhere in $\mathbb{R}^2$ outside some set Σ_f of isolated points and differs from an affine function.

In the case when the space-like condition (2) holds everywhere in $\mathbb{R}^2 \setminus \Sigma_f$, the results by Ecker [3] and Klyachin and Miklyukov [5] imply that in a neighborhood of each point $a \in \Sigma_f$ the solution $f(x,y)$ has the same asymptotic structure as the light cone. In other words, for $\varepsilon^2 = 1$ we have the expansion

$$f(w) = f(a) + \varepsilon\|w - a\| + \bar{o}(\|w - a\|), \quad w \to a, \tag{4}$$

where $w = (x,y)$.

It is unknown whether (4) holds in the general case for an arbitrary almost entire C^2-solution to (1). We note that the minimal surfaces in $\mathbb{R}^3$ have no isolated singularities, as it follows from the results about removable singularities for two-dimensional minimal surfaces [4, 6].

We also note that there is a certain analog of the Weierstrass representation for solutions to (3). This makes it possible to parametrize solutions locally via holomorphic functions. In particular, such solutions are analytic functions outside the singular set. There is no analogous representation in the general case of equation (1). This follows from the mixed elliptic-parabolic type of equation (1) and the existence of solutions to (1) of a low smoothness class. Indeed, every function of the form

$$f(x,y) = x + h(y) \tag{5}$$

satisfies (1) under the only condition that $h \in C^2(\mathbb{R})$. Moreover, it is obvious that $|\nabla f|^2 = 1 + h'^2(y) \geq 1$ everywhere in $\mathbb{R}^2$.

1.2. Most articles devoted to equation (1) deal only with the space-like case (2) in which the equation is of elliptic type. An important but almost uninvestigated case is that of mixed solutions, i.e. solutions to (1) for which there are points at which $|\nabla f(x,y)| < 1$ as well as points at which $|\nabla f(x,y)| > 1$. In this article we, for the first time, construct families of such mixed-type solutions which are bounded and analytic as functions in (x,y) outside the set of singularities. It is worth to note that the singularities of these solutions also have the asymptotic structure of the light cone.

In view of the above-mentioned circumstances, we use a method that differs essentially from the Weierstrass representation and allows us to construct doubly periodic solutions to equation (1) in nondivergent form, i.e. solutions satisfying

$$f(x+\tau_1 n, y+\tau_2 m) = f(x,y)$$

for some $\tau_1, \tau_2 > 0$ and arbitrary integers n and m. Under certain assumptions, the solution $f(x,y)$ has isolated singularities at the nodes of the rectangular lattice $\Sigma_f = \tau_1\mathbb{Z} \oplus \tau_2\mathbb{Z}$. The structure and functional-geometric properties of these solutions depend on the so-called generating matrix A whose definition is given in the next section.

We do not dwell upon the case of periodic solutions. They are the topic of the recent article [7], wherein examples were exhibited of periodic almost entire maximal surfaces with isolated singular points on a straight line. These examples result from one general assertion of this article in which the authors completely classified solutions to (1) with the so-called "harmonic" level lines.

Theorem 1. *Let $f(x,y)$ be a C^2-solution to (1) of the form $F\circ\varphi(x,y)$, where $\varphi(x,y)$ is a harmonic function and F is a function of the smoothness class $C^2(\mathbb{R})$. Then*

$$\varphi(x,y) = \operatorname{Re}\int \frac{d\zeta}{g(\zeta)},$$

where $g(\zeta)$ is one of the following functions:

1) $g(\zeta) = a\zeta + c$;
2) $g(\zeta) = a\,e^{b\zeta}$;
3) $g(\zeta) = a\sin(b\zeta + c)$.

Here $\zeta = x + iy \in \mathbb{C}$, $a^2, b^2 \in \mathbb{R}$, and $c \in \mathbb{C}$. The function $F(t)$ is a solution to the functional-differential equation $F''(t) + \xi(F(t)) = 0$, with the real function $\xi(F)$ soundly defined from the relation

$$\xi(\varphi(x,y)) + \frac{1}{|g(\zeta)|^2}\operatorname{Re} g'(\zeta) = 0.$$

The cases 1 and 2 give classical examples of maximal surfaces: planes, maximal catenoids, helicoids, and maximal Scherk surfaces. In the case 3 the surface is space-like only if $a, b \in \mathbb{R}$; then there are infinitely many isolated singular points on a straight line. An implicit representation for this surface up to translation and homothety of the (x, y, z) space is as follows

$$\operatorname{sn}\left(\frac{z}{k'}; k\right) = \frac{\cos x}{\operatorname{ch} y},$$

where $k' = \sqrt{1-k^2}$, $k \in (0,1)$, and $\operatorname{sn}(t; k)$ is the Jacobi sine defined in Section 6.

2. Generating matrices

2.1. Throughout the sequel, by a matrix we mean a square 3×3-matrix with real entries. The set of permutations of the index set $\{1, 2, 3\}$ is denoted by S. In this section we define the notion of generating matrix and consider its properties.

Definition 2. A matrix $A = (a_{ij})$, not identically equal to zero, is a *generating matrix*, or $A \in M$, if

$$a_{i\alpha}a_{i\beta} = a_{j\gamma}a_{k\gamma} \tag{6}$$

for arbitrary permutations $(i, j, k), (\alpha, \beta, \gamma) \in S$.

The *associate matrix* A' of A is defined as follows:

$$A' = \begin{pmatrix} a_{11} & a_{22} & a_{33} \\ a_{23} & a_{31} & a_{12} \\ a_{32} & a_{13} & a_{21} \end{pmatrix}. \tag{7}$$

Obviously, the generation condition (6) amounts to vanishing of all second order minors of the matrix A'. Since A is a nonzero matrix, A is a generating matrix if and only if $\operatorname{rank} A' = 1$. This means in particular that every generating matrix A can be written down as

$$A = \begin{pmatrix} p_1p_2 & q_1r_2 & r_1q_2 \\ r_1r_2 & p_1q_2 & q_1p_2 \\ q_1q_2 & r_1p_2 & p_1r_2 \end{pmatrix}, \tag{8}$$

where $p_i^2 + q_i^2 + r_i^2 \neq 0$, $i = 1, 2$. Indeed, since $\operatorname{rank} A' = 1$, there are a nonzero vector $\xi = (p_1, q_1, r_1)$ and numbers p_2, q_2, and r_2, not vanishing simultaneously, such that the columns of A' are proportional to ξ with the respective coefficients p_2, q_2, and r_2; hence, (8) follows.

We point to a useful simple consequence of the generation condition: the product of the entries in one column or in one row of a generating matrix always equals the same number. The latter is called the *modulus* of A and denoted by $\theta(A)$. If $\theta(A) \neq 0$ then we say that A is *nondegenerate*.

Lemma 3. *If $A \in M$ is degenerate, i.e., $\theta(A) = 0$, then after a suitable permutation of rows and columns A takes one of the following forms:*

$$D_1 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{pmatrix},$$
$$D_2 = \begin{pmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix}, \tag{9}$$
$$D_3 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix},$$

with $a_{ij} \neq 0$, or A has a zero row or a zero column.

Proof. Using the above representation (8) for a generating matrix, we come to the sought result by nullifying by turns the components of the vectors (p_1, q_1, r_1) and (p_2, q_2, r_2). □

Lemma 4. *The quantity*

$$\Delta_\alpha := (a_{j\alpha} + a_{k\alpha} - a_{i\alpha})^2 - 4a_{i\beta}a_{i\gamma}, \tag{10}$$

where $(i, j, k), (\alpha, \beta, \gamma) \in S$, depends only of the index α.

Proof. From the generation relations we obtain

$$\begin{aligned}\Delta_\alpha &= a_{j\alpha}^2 + a_{k\alpha}^2 + a_{i\alpha}^2 + 2(a_{j\alpha}a_{k\alpha} - 2a_{i\beta}a_{i\gamma} - a_{j\alpha}a_{i\alpha} - a_{k\alpha}a_{i\alpha}) \\ &= a_{j\alpha}^2 + a_{k\alpha}^2 + a_{i\alpha}^2 - 2(a_{j\alpha}a_{k\alpha} + a_{j\alpha}a_{i\alpha} + a_{k\alpha}a_{i\alpha}) \\ &= \sum_{\sigma=1}^{3} a_{\sigma\alpha}^2 - \sum_{\sigma\neq\rho} a_{\sigma\alpha}a_{\rho\alpha}.\end{aligned}$$

The last expression depends only on α. □

Below we use the quantity $\Delta := \frac{1}{4}\Delta_2$, calling it the *discriminant* of A.

2.2. Let $\mathbb{R}^+ \otimes \mathbb{R}^+$ be the direct product of two multiplicative groups of positive real numbers which is endowed with the conventional multiplication; i.e., $\mathbb{R}^+ \otimes \mathbb{R}^+$ is furnished with the operation $\overline{\lambda} * \overline{\mu} = (\lambda_1\mu_1, \lambda_2\mu_2)$, where $\overline{\lambda} = (\lambda_1, \lambda_2)$ and $\overline{\mu} = (\mu_1, \mu_2)$. Define the action of this group on the set of all matrices as follows:

$$\overline{\lambda}(A) = \begin{pmatrix} \frac{1}{\lambda_1}a_{11} & a_{12} & \lambda_1 a_{13} \\ \frac{1}{\lambda_2}a_{21} & a_{22} & \lambda_2 a_{23} \\ \lambda_1\lambda_2 a_{31} & a_{32} & \frac{1}{\lambda_1\lambda_2}a_{33} \end{pmatrix}. \tag{11}$$

It is easy to see that if $A \in M$ then also $\overline{\lambda}(A) \in M$; moreover, $\theta(\overline{\lambda}(A)) = \theta(A)$. From the viewpoint of further consideration, the matrices $\overline{\lambda}(A)$ and A generate the same surface; therefore, it is convenient to identify them.

Definition 5. A matrix D is said to be $\overline{\lambda}$-*equivalent* to a matrix A if $D=\overline{\lambda}(A)$ for some $\overline{\lambda}=(\lambda_1,\lambda_2)\in\mathbb{R}^+\otimes\mathbb{R}^+$.

It is easy to verify that $\overline{\lambda}$-equivalence is in fact an equivalence relation.

Lemma 6. *Let $A\in M$ be a generating matrix whose modulus differs from zero: $\theta(A)\neq 0$. Then A is $\overline{\lambda}$-equivalent to a matrix of the form*

$$\begin{pmatrix} a & b & c\\ \varepsilon_2\varepsilon_3 c & \varepsilon_2 a & \varepsilon_3 b\\ \varepsilon_2\varepsilon_3 b & \varepsilon_2 c & \varepsilon_3 a\end{pmatrix},\quad \varepsilon_2^2=\varepsilon_3^2=1. \tag{12}$$

Proof. Put $\varepsilon_k=\operatorname{sign}(a_{kk}/a_{11})$, $k=2,3$, and choose $\lambda_1=\varepsilon_2a_{11}/a_{22}$ and $\lambda_2=\varepsilon_3a_{33}/a_{11}$. Then $\lambda_i>0$ and, setting $a=\varepsilon_2a_{22}$, $b=a_{12}$, and $c=\varepsilon_3a_{32}$, we conclude that $\overline{\lambda}(A)$ has the form (12), as desired. □

Remark 7. It is easy to see that the matrices (12) are pairwise $\overline{\lambda}$-nonequivalent for distinct choices of $(\varepsilon_2,\varepsilon_3)$.

3. Doubly periodic solutions

3.1. To construct doubly periodic solutions to (1), we consider the special class of solutions $z(x,y)$ defined implicitly as follows:

$$\zeta(z(x,y))=\varphi(x)\psi(y), \tag{13}$$

where φ, ψ, and ζ are some C^2-smooth functions. We use relation (13) to transform equation (1). We have

$$\begin{aligned}\zeta'(z)z'_x&=\varphi'\psi,\\ \zeta'(z)z'_y&=\varphi\psi'\end{aligned} \tag{14}$$

and

$$\begin{aligned}\zeta''z_x^2+\zeta'z_{xx}&=\varphi''\psi,\\ \zeta''z_xz_y+\zeta'z_{xy}&=\varphi'\psi',\\ \zeta''z_y^2+\zeta'z_{yy}&=\varphi\psi''.\end{aligned} \tag{15}$$

Multiplying the last three expressions by

$$\begin{aligned}(1-z_y^2)\zeta'^2&\equiv\zeta'^2-\varphi^2\psi'^2,\\ 2z_xz_y\zeta'^2&\equiv 2\varphi\psi\varphi'\psi',\\ (1-z_x^2)\zeta'^2&\equiv\zeta'^2-\varphi'^2\psi^2\end{aligned}$$

respectively and summing the results, in view of (1) we obtain

$$\zeta''\zeta'^2(z_x^2+z_y^2)=\zeta'^2(\varphi''\psi+\varphi\psi'')-\varphi\psi(\varphi\varphi''\psi'^2-2\varphi'^2\psi'^2+\varphi'^2\psi\psi'').$$

Using (14) on the left-hand side together with equality (13), we finally derive that

$$\zeta'^2(\varphi''\psi+\varphi\psi'')-\zeta(\varphi\varphi''\psi'^2-2\varphi'^2\psi'^2+\varphi'^2\psi\psi'')-\zeta''(\varphi'^2\psi^2+\varphi^2\psi'^2)=0. \tag{16}$$

Lemma 8. *Let $z(x,y)$ be a solution to (1) and let φ, ψ, and ζ be functions satisfying (13) and such that*

(i) *φ and ψ have zeros;*

(ii) *there are functions P, Q, and H for which*

$$\varphi'^2 = P(\varphi^2),\qquad \psi'^2 = Q(\psi^2),\qquad \zeta'^2 = H(\zeta^2). \tag{17}$$

Then P and Q are polynomials of degree at most 2.

Proof. From (17) we find

$$\varphi'' = P'(\varphi^2)\varphi,\qquad \psi'' = Q'(\psi^2)\psi,\qquad \zeta'' = H'(\zeta^2)\zeta.$$

Together with (16), this gives

$$\zeta H(\zeta^2)\big[P'(\varphi^2) + Q'(\psi^2)\big] - \zeta\Big(\varphi^2 P'(\varphi^2)Q(\psi^2) - 2P(\varphi^2)Q(\psi^2) + \psi^2 Q'(\psi^2)P(\varphi^2)\Big) - \zeta H'(\zeta^2)\big[\psi^2 P(\varphi^2) + \varphi^2 Q(\psi^2)\big] = 0.$$

Differentiating the last equality with respect to ζ and nullifying φ and ψ by turns, we come to the equalities

$$\begin{aligned}(h_0 - p_0 v)Q'(v) + 2p_0 Q(v) + h_0 p_1 - h_1 p_0 v = 0,\\ (h_0 - q_0 u)P'(u) + 2q_0 P(u) + h_0 q_1 - h_1 q_0 u = 0,\end{aligned} \tag{18}$$

where $p_0 = P(0)$, $p_1 = P'(0)$, $q_0 = Q(0)$, $q_1 = Q'(0)$, $h_0 = H(0)$, and $h_1 = H'(0)$. For $p_0 = 0$ ($q_0 = 0$), this implies that $Q(v)$ $\big(P(u)\big)$ is a linear function. Otherwise, solving the ordinary differential equations (18), we have

$$P(u) = au^2 + \left(h_1 - 2a\frac{h_0}{q_0}\right)u - \frac{1}{2q_0}\left(h_0(h_1 + q_1) - 2a\frac{h_0^2}{q_0}\right),$$

$$Q(v) = bv^2 + \left(h_1 - 2b\frac{h_0}{p_0}\right)v - \frac{1}{2p_0}\left(h_0(h_1 + p_1) - 2b\frac{h_0^2}{p_0}\right),$$

with $a, b \in \mathbb{R}$ arbitrary numbers. This completes the proof of the lemma. □

Note that the absence of condition (i) essentially enlarges the class of solutions to the functional-differential equation (16) owing to the existence of, for example, radially symmetric solutions and solutions of the form (5). This means in particular that solutions may be unbounded and be of a low smoothness class.

3.2. Define the functions φ, ψ, and ζ in (13) by the equations

$$\begin{aligned} \varphi'^2 &= a_1 - 2b_1\varphi^2 + c_1\varphi^4, \\ \psi'^2 &= a_2 - 2b_2\psi^2 + c_2\psi^4, \\ \zeta'^2 &= c_3 + 2b_3\zeta^2 + a_3\zeta^4, \end{aligned} \tag{19}$$

where a_i, b_i, and c_i $(i = 1, 2, 3)$ are some real numbers. We demonstrate that to each solution of (1) satisfying (13) and (19) there corresponds some generating matrix $A \in M$. Observe that

$$\begin{aligned} \varphi'' &= 2\varphi(c_1\varphi^2 - b_1), \\ \psi'' &= 2\psi(c_2\psi^2 - b_2), \\ \zeta'' &= 2\zeta(a_3\zeta^2 + b_3). \end{aligned} \tag{20}$$

Inserting (19) and (20) in (16), we obtain

$$\zeta^4(c_1c_2 - a_3b_1 - a_3b_2) + \zeta^2\Big[(a_3a_2 - c_1b_3 - c_1b_2)\varphi^2 + (a_3a_1 - b_1c_2 - c_2b_3)\psi^2\Big]$$
$$+ (a_1b_3 + a_1b_2 - c_2c_3)\psi^2 + (a_2b_1 - c_1c_3 + a_2b_3)\varphi^2 + b_2c_3 + b_1c_3 - a_1a_2 = 0.$$

By independence of the functions $\varphi(x)$ and $\psi(y)$, all coefficients of φ^2, ψ^2, and ζ^4 in the last equality vanish. Putting $\beta_i = b_j + b_k$ for an arbitrary permutation $(i, j, k) \in S$, we arrive at the generation conditions for the matrix

$$A = \begin{pmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{pmatrix}. \tag{21}$$

We have thus proven the following

Theorem 9. *The function $z(x, y)$ defined implicitly by the relation*

$$\zeta\big(z(x, y)\big) = \varphi(x)\psi(y),$$

where φ, ψ, and ζ satisfy (19), *is a solution to* (1) *if and only if the matrix A in* (21) *is generating.*

Remark 10. Choosing the entries of the matrix A in (21) so that the elliptic functions φ and ψ of (19) are periodic, we obtain doubly periodic solutions to (1).

We list some properties of the discriminant Δ of a matrix $A \in M$ which was introduced in Subsection 2.1.

Lemma 11. *The discriminant Δ of the generating matrix* (21) *has the following representation*:

$$\Delta \equiv b_i^2 - a_i c_i = -(b_1 b_2 + b_3 \beta_3),$$

where $b_i = \frac{1}{2}(\beta_j + \beta_k - \beta_i)$ for arbitrary $(i,j,k) \in S$; in particular, it is independent of $i = 1,2,3$.

Proof. The sought identity is immediate from the definition of the discriminant Δ. Using the generation conditions of the matrix (21), we easily arrive at the sought representation

$$\Delta = b_1^2 - a_1 c_1 = b_1^2 - \beta_2\beta_3 = b_1^2 - (b_1 + b_3)\beta_3 = b_1(b_1 - \beta_3) - b_3\beta_3 = -b_1 b_2 - b_3\beta_3.$$

In what follows, we only consider solutions to (1) generated by the matrices $A \in M$.

4. Singularities of doubly periodic solutions

4.1. Let $f(x,y)$ be a C^2-solution to (1) in the domain $\mathbb{R}^2 \setminus \Sigma_f$, where Σ_f is a set of isolated points. If $f(x,y)$ does not extend to a C^2-solution in some neighborhood of a point $m_0 = (x_0, y_0) \in \Sigma_f$, then we call m_0 a *singular point* (*singularity*) of the solution $f(x,y)$. Clearly, one of the following conditions is necessary for $m_0 = (x_0, y_0)$ to be a singular point of a solution $z(x,y)$ defined implicitly by the relation $F(x, y, z(x,y)) = 0$: (i) $\nabla F(m_0) = 0$; (ii) $\nabla F(m_0)$ does not exist. Note that, for the solutions satisfying (13) and (19), the condition $\nabla F(m_0) = 0$ can be written down as

$$\begin{aligned} (a_1 - 2b_1\varphi_0^2 + c_1\varphi_0^4)\psi_0^2 &= 0,\\ (a_2 - 2b_2\psi_0^2 + c_2\psi_0^4)\varphi_0^2 &= 0,\\ c_3 + 2b_3\zeta_0^2 + a_3\zeta_0^4 &= 0, \end{aligned} \tag{22}$$

where $\varphi_0 = \varphi(x_0)$, $\psi_0 = \psi(y_0)$, and $\zeta_0 = \varphi_0\psi_0$.

Theorem 12. *Let $m_0 = (x_0, y_0)$ be an isolated singular point of a solution $z(x,y)$ to* (1) *corresponding to some generating matrix $A \in M$. Then the discriminant Δ is greater than* 0 *and the following equalities hold*:

$$\begin{aligned} \varphi^2(x_0) &= (b_1 + \delta\sqrt{\Delta})/c_1,\\ \psi^2(y_0) &= (b_2 + \delta\sqrt{\Delta})/c_2, \end{aligned}$$

where $\delta^2 = 1$. In particular, $\varphi'(x_0) = \psi'(y_0) = \zeta'(z_0) = 0$.

Proof. Put $F(x,y,z(x,y)) = \varphi(x)\psi(y) - \zeta(z(x,y))$. Then $\nabla F = \{\varphi'\psi, \varphi\psi', -\zeta'\}$, and it is easy to see that $\nabla F(m_0) = 0$, since m_0 is an isolated singular point.

First, suppose that $\Delta < 0$. Since $\Delta = b_3^2 - a_3c_3 < 0$, it follows that $c_3 + 2b_3\zeta_0^2 + a_3\zeta_0^4 \neq 0$, and equations (22) then imply $\nabla F(m_0) \neq 0$; a contradiction. Hence, $\Delta \geq 0$. In the case of $\Delta = 0$ we have trivial solutions without singular points: linear functions. Thus, $\Delta > 0$.

We demonstrate that $\varphi(x_0)\psi(y_0) \neq 0$. Assume the contrary; for example, let $\varphi(x_0) = 0$. Then from the last equation of (22) we have $c_3 = 0$, since $\zeta_0 = \varphi_0\psi_0$. The following two cases are possible: $a_1 \neq 0$ and $a_1 = 0$. Suppose that $a_1 \neq 0$. Then it follows from (22) that $\psi(y_0) = 0$; moreover, $\zeta'^2 = 2b_3\zeta^2 + a_3\zeta^4$, $\zeta(z_0) = 0$. Hence, $1/\zeta(z)$ equals one of the functions $\sin(\mu z+\lambda)$, $\sinh(\mu z+\lambda)$, or $\cosh(\mu z+\lambda)$. Therefore, it has a zero $z_0 \in \mathbb{R}$, which is impossible. Consider the second case, $a_1 = 0$. From the first equation of (22) we infer that (x_0, y) is a singular point for every $y \in \mathbb{R}$; i.e., $m_0 = (x_0, y_0)$ is not an isolated singular point. This contradiction with the hypothesis of the theorem implies that $\varphi(x_0)\psi(y_0) \neq 0$. In particular, from (22) and (19) we have $\varphi'(x_0) = \psi'(y_0) = 0$. Using the fact that the discriminant Δ is greater than 0, from the first two equations of (22) we find

$$\varphi^2(x_0) = (b_1 + \delta\sqrt{\Delta})/c_1,$$
$$\psi^2(y_0) = (b_2 + \delta\sqrt{\Delta})/c_2,$$

with $\delta^2 = 1$. To complete the proof of the theorem, it suffices to check the fulfillment of the third equation in (22) for $\zeta_0 = \varphi(x_0)\psi(y_0)$. From the generation conditions of the matrix A we have

$$\begin{aligned}\varphi_0\psi_0 &= \frac{1}{c_1c_2}(b_1 + \delta\sqrt{\Delta})(b_2 + \delta\sqrt{\Delta}) \\ &= \frac{1}{a_3\beta_3}(b_1b_2 + \beta_3\delta\sqrt{\Delta} + \Delta) \\ &= \frac{1}{a_3\beta_3}(b_1b_2 + \Delta) + \frac{1}{a_3}\delta\sqrt{\Delta}.\end{aligned}$$

Using the representation of Lemma 11, we finally obtain

$$\zeta_0^2 = (-b_3 + \delta\sqrt{\Delta})/a_3.$$

It is easy to verify that the found value ζ_0 is really a root of the last equation in (22), which completes the proof of the theorem. □

4.2. Suppose that a solution $z(x,y)$ to (1) generated by a matrix $A \in M$ is defined implicitly as $\zeta(z(x,y)) = \varphi(x)\psi(y)$. Denote by F the surface that is defined by this solution. In the next theorem we prove that, in a sufficiently small neighborhood about an isolated singular point, the surface F behaves asymptotically like the light cone.

Theorem 13. *Let $m_0 = (x_0, y_0)$ be an isolated singular point of a solution $z(x,y)$ to (1) generated by a matrix $A \in M$. Then the following expansion holds:*

$$z(m) = z_0 + \delta\|m - m_0\| + \bar{o}(\|m - m_0\|), \quad m \to m_0,$$

where $\delta^2 = 1$, $m = (x,y)$, and $z_0 = z(x_0, y_0)$.

Proof. By the Taylor formula for (13) we have

$$\begin{aligned} &\zeta_0 + \zeta_0'(z - z_0) + \frac{1}{2}\zeta_0''(z - z_0)^2 \\ &\quad = \varphi_0\psi_0 + \psi_0\varphi_0'(x - x_0) + \varphi_0\psi_0'(y - y_0) \\ &\qquad + \frac{1}{2}\Big(\psi_0\varphi_0''(x - x_0)^2 + \varphi_0\psi_0''(y - y_0)^2 + \varphi_0'\psi_0'(x - x_0)(y - y_0)\Big) \\ &\qquad + \bar{o}(\|m' - m_0'\|^2), \end{aligned} \tag{23}$$

where $m_0' = (x_0, y_0, z_0)$, $m' = (x, y, z) \to m_0'$, and $\varphi_0 = \varphi(x_0)$, $\psi_0 = \psi(y_0)$, $\zeta_0 = \zeta(z_0)$. Since m_0 is an isolated singular point, the preceding theorem implies that $\varphi'(x_0) = \psi'(y_0) = \zeta'(z_0) = 0$ and moreover $\zeta(z_0) = \varphi(x_0)\psi(y_0)$. Simplifying (23), we then obtain

$$A(z - z_0)^2 = B(x - x_0)^2 + C(y - y_0)^2 + \bar{o}(\|m' - m_0'\|), \quad m' \to m_0',$$

where $A = \zeta_0''$, $B = \psi_0\varphi_0''$, and $C = \varphi_0\psi_0''$. The theorem will be proven if we demonstrate that $A = B = C \neq 0$. Taking (20) into account, we find that

$$\begin{aligned} A &= \zeta_0'' = 2\zeta_0(a_3\zeta_0^2 + b_3), \\ B &= \psi_0\varphi_0'' = 2\zeta_0(c_1\varphi_0^2 - b_1), \\ C &= \varphi_0\psi_0'' = 2\zeta_0(c_2\psi_0^2 - b_2). \end{aligned} \tag{24}$$

By Theorem 12 we have

$$\begin{aligned} \varphi_0^2 &= \big(b_1 + \delta\sqrt{\Delta}\big)/c_1, \\ \psi_0^2 &= \big(b_2 + \delta\sqrt{\Delta}\big)/c_2, \\ \zeta_0^2 &= \big(-b_3 + \delta\sqrt{\Delta}\big)/a_3, \end{aligned}$$

where $\delta^2 = 1$. Inserting the last three relations in (24), we obtain

$$A = B = C = 2\zeta_0\delta\sqrt{\Delta} \neq 0.$$

We have thus verified that the surface F near an isolated singular point exhibits a behavior like that of the light cone. □

5. Classification of solutions with isolated singularities

5.1. Here we study the behavior of the gradient of a solution $z(x,y)$ to (1), generated by a matrix $A \in M$, in infinitely small neighborhoods $O(m)$ of isolated singular points m. It turns out that the surfaces F defined by these solutions can be split into the following three types depending on the behavior of the surfaces in the neighborhoods of singular points:

1) space-like $\big(|\nabla z| < 1\big)$ everywhere in the neighborhood $O(m)$;

2) the neighborhood $O(m)$ includes two alternating connected components of the space-like domain and the time-like domain for the surface F;

3) the neighborhood $O(m)$ includes four alternating connected components of the space-like domain and the time-like domain for the surface F.

We linearize the level sets $\big|\nabla z(x,y)\big| = 1$ in the infinitely small neighborhood $O(m)$ of an isolated singular point $m = (x_0, y_0)$ for a solution $z(x,y)$ to (1) generated by a matrix $A \in M$. Since $|\nabla z|^2 = (\varphi^2\psi'^2 + \varphi'^2\psi^2)/\zeta'^2$; therefore, $|\nabla z| = 1$ if and only

$$\varphi'^2\psi^2 + \varphi^2\psi'^2 - \zeta'^2 = 0. \tag{25}$$

We now find expansions of the functions φ and ψ and their derivatives. We do this in the general form, defining the function $g(u)$ by the relation $g'^2 = P(g^2)$, where $P(t) = a + 2bt + ct^2$. Put $g_0 = g(u_0)$ and observe that by Theorem 12 $g_0^2 = \big(-b + \delta\sqrt{\Delta}\big)/c$, with $\Delta = b^2 - ac$. Therefore,

$$g_0'' = 2(b + cg_0^2)g_0 = 2\delta\sqrt{\Delta}g_0 = 2\mu g_0,$$

where $\mu = \delta\sqrt{\Delta}$. For the third and fourth derivatives of g at u_0 we have

$$g_0^{(3)} = 2g_0'(b + 3cg_0^2) = 0,$$

since $g_0' = 0$, and

$$g_0^{(4)} = 2g_0''(b + 3cg_0^2) + 2g_0'(b + 3cg_0^2)' = 2g_0''(b + 3cg_0^2) = 4\mu g_0(b + 3cg_0^2).$$

Thus, $g = g_0 + \mu g_0 h^2 + \frac{1}{6}g_0\mu(b + 3cg_0^2)h^4 + \bar{o}(h^4)$, where $h = u - u_0$, and

$$g^2 = g_0^2\left(1 + 2\mu h^2 + h^4\left(\mu^2 + \frac{1}{3}\mu(b + 3cg_0^2)\right)\right) + \bar{o}(h^4). \tag{26}$$

For the derivative of g we obtain the expansion

$$g'^2 = P(g_0^2) + P'(g_0^2)\rho + \frac{1}{2}P''(g_0^2)\rho^2 + \bar{o}(\rho^2), \tag{27}$$

with $\rho \equiv g^2 - g_0^2$. Calculating $P(g_0^2) = 0$, $P'(g_0^2) = 2\mu$, $P''(g_0^2) = 2c$ and applying (26), we deduce

$$\begin{aligned} g'^2 &= 2\mu\left(2\mu g_0^2 h^2 + h^4 g_0^2\left(\mu^2 + \frac{1}{3}\mu(b + 3cg_0^2)\right)\right) + 4cg_0^4\mu^2 h^4 + \bar{o}(h^4) \\ &= 4\mu^2 g_0^2 h^2 + h^4 g_0^2\left(2\mu^3 + \frac{2}{3}\mu^2(b + 3cg_0^2) + 4\mu^2 cg_0^2\right) + \bar{o}(h^4) \\ &= 4\mu^2 g_0^2 h^2 + h^4 g_0^2\left(2\mu^3 + \frac{2}{3}\mu^2(b + 9cg_0^2)\right) + \bar{o}(h^4). \end{aligned}$$

Note that $b + 9cg_0^2 = -8b + 9\delta\sqrt{\Delta} = 9\mu - 8b$. Therefore,

$$g'^2 = 4\mu^2 g_0^2 h^2\left(1 + 2h^2\left(\mu - \frac{2}{3}b\right)\right) + \bar{o}(h^4). \tag{28}$$

Substituting the found expansions (26) and (28) for the concrete functions φ, ψ, and ζ in equation (25), we obtain

$$\begin{aligned} &\varphi_0^2(1 + 2\mu h_x^2)4\mu^2\psi_0^2 h_y^2\left(1 + 2h_y^2\left(\mu + \frac{2}{3}b_2\right)\right) \\ &\quad + \psi_0^2(1 + 2\mu h_y^2)4\mu^2\varphi_0^2 h_x^2\left(1 + 2h_x^2\left(\mu + \frac{2}{3}b_1\right)\right) \\ &\quad - 4\mu^2\zeta_0^2 h_z^2\left(1 + 2h_z^2\left(\mu - \frac{2}{3}b_3\right)\right) = 0. \end{aligned} \tag{29}$$

Since by Theorem 13 the surface F in the neighborhood $O(m)$ behaves like the light cone; therefore, $h_z^2 = h_x^2 + h_y^2$. Simplifying equation (29), we hence deduce that

$$b_3(h_x^2 + h_y^2)^2 + b_2 h_y^4 + b_1 h_x^4 = 0.$$

Setting $\xi = h_y^2/h_x^2$, we obtain $b_3(1+\xi)^2 + b_2\xi^2 + b_1 = 0$, or

$$\beta_1\xi^2 + 2b_3\xi + \beta_2 = 0, \tag{30}$$

where $\beta_i = b_j + b_k$ for arbitrary permutations $(i, j, k) \in S$.

Observe that the discriminant D of the quadratic equation (30) is positive, because $D = 4(b_3^2 - \beta_1\beta_2) = 4\Delta > 0$. We thus obtain the following linearization of the level sets $|\nabla z(x, y)| = 1$:

$$(h_y^2 - \xi_1 h_x^2)(h_y^2 - \xi_2 h_x^2) = 0, \tag{31}$$

where ξ_1 and ξ_2 are roots of the quadratic equation (30). Depending on the sign of ξ_1 and ξ_2, we come to the sought classification of the surfaces generated by matrices $A \in M$ and having isolated singular points:

1st type	2nd type	3rd type
$\beta_1 > 0,\ \beta_2 > 0,\ b_3 > 0$	$\beta_1 < 0,\ \beta_2 > 0$	$\beta_1 > 0,\ \beta_2 > 0,\ b_3 < 0$
$\beta_1 < 0,\ \beta_2 < 0,\ b_3 < 0$		$\beta_1 < 0,\ \beta_2 < 0,\ b_3 > 0$

6. Examples of construction of doubly periodic solutions

We now exhibit examples of the surfaces generated by matrices $A \in M$ for each of the three types in the above classification. Given an arbitrary real $k \in (0,1)$, we define the conjugate number $k' = \sqrt{1-k^2}$. We denote by $\operatorname{sn}(t;k)$ the Jacobi sine with parameter k which is defined by the equality

$$\int_0^{\operatorname{sn}(t;k)} \frac{du}{\sqrt{(1-u^2)(1-k^2u^2)}} = t,$$

and denote by $\operatorname{cn}(t;k)$ the Jacobi cosine which satisfies the relation $\operatorname{cn}(t;k) = \sqrt{1-\operatorname{sn}^2(t;k)}$. We note that the discriminant of the matrix A equals $\Delta \equiv 1/4$ in all examples to be exhibited below.

1st type:

$$\operatorname{sn}(\lambda z; km) = \operatorname{cn}\left(x; \frac{k}{\sqrt{1+k^2}}\right) \operatorname{cn}\left(y; \frac{m}{\sqrt{1+m^2}}\right),$$

where $\lambda = 1/(km)'$, $k, m > 0$, and $0 < km < 1$. The generating matrix has the form

$$\begin{pmatrix} \frac{1}{1+k^2} & -\frac{(1+k^2)m^2\lambda^2}{(1+m^2)} & -\frac{k^2}{1+k^2} \\ \frac{1}{1+m^2} & -\frac{(1+m^2)k^2\lambda^2}{(1+k^2)} & -\frac{m^2}{1+m^2} \\ k^2m^2\lambda^2 & \frac{1}{\lambda^2(1+k^2)(1+m^2)} & \lambda^2 \end{pmatrix}.$$

2nd type:

$$\operatorname{cn}(z; \mu km) = \operatorname{sn}\left(\frac{x}{k'}; k\right) \operatorname{cn}(y; m),$$

where $\mu = 1/(k'm)'$, $k, m \in (0,1)$. The generating matrix is

$$\begin{pmatrix} \frac{1}{k'^2} & -(k'm'm\mu)^2 & \frac{k^2}{k'^2} \\ m'^2 & \frac{k^2\mu^2}{k'^2} & -m^2 \\ -k^2m^2\mu^2 & \frac{1}{k'^2\mu^2} & m'^2\mu^2 \end{pmatrix}.$$

3rd type:

$$\operatorname{sn}(\lambda z; km) = \operatorname{sn}\left(\frac{x}{k'}; k\right) \operatorname{sn}\left(\frac{y}{m'}; m\right),$$

where $\lambda = 1/(km)'$, $k, m \in (0,1)$. The generating matrix is

$$\begin{pmatrix} \frac{1}{k'^2} & \frac{\lambda^2 k'^2 m^2}{m'^2} & \frac{k^2}{k'^2} \\ \frac{1}{m'^2} & \frac{\lambda^2 k^2 m'^2}{k'^2} & \frac{m^2}{m'^2} \\ \lambda^2 k^2 m^2 & \frac{1}{\lambda^2 k'^2 m'^2} & \lambda^2 \end{pmatrix}.$$

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